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Heat transfer in a liquid film on an unsteady stretching surface

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Abstract

The momentum and heat transfer in a laminar liquid film on a horizontal stretching sheet is analysed. The governing time-dependent boundary layer equations are reduced to a set of ordinary differential equations by means of an exact similarity transformation. The resulting two-parameter problem is solved numerically for some representative values of the unsteadiness parameter S for Prandtl numbers from 0.001 to 1000. The temperature is observed to increase monotonically from the elastic sheet towards the free surface except in the high diffusivity limit $Pr \to 0$ where the surface temperature approaches that of the sheet. A low stretching rate, i.e. high values of S, tends to reduce the surface temperature for all Prandtl numbers. The heat flux from the liquid to the elastic sheet decreases with S for $Pr \lesssim 0.1$ and increases with increased unsteadiness for $Pr \gtrsim 1$.

1. Introduction

The analysis of fluid flow and heat transfer across a thin liquid film is important for the understanding and design of various heat exchangers and chemical processing equipment. Applications include wire and fiber coating, food stuff processing, reactor fluidization, transpiration cooling, polymer processing, etc. Production of a thin liquid film either on the surface of a vertical wall by means of the action of gravity or on a rotating horizontal disk due to the action of centrifugal forces has been studied considerably in the literature (see e.g. Sparrow and Gregg [1,2] and Dandapat and Ray [3,4]). In a melt-spinning process, the extrudate from the die is generally drawn and simultaneously stretched into a filament or sheet, which is then solidified through rapid quenching or gradual cooling by direct contact with water or chilled metal rolls. In fact, stretching imports a unidirectional orientation to the extrudate, thereby improving its mechanical properties and the quality of the final product greatly depends on the rate of cooling.

A class of flow problems with obvious relevance to polymer extrusion is the flow induced by the stretching motion of a flat elastic sheet. Crane [5] was the first who studied the motion set up in the ambient fluid due to a linearly stretching surface. Several authors, e.g. Refs. [6–14], have subsequently explored various aspects of the accompanying heat transfer occurring in the infinite fluid medium surrounding the stretching sheet. The hydrodynamics of a finite fluid medium, i.e. a thin liquid film, on a stretching sheet was first con-

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Nomenclature

b	stretching rate [s^{-1}]
c	constant, Eq. (14b)
c_p	specific heat [J kg^{-1} K^{-1}]
f	dimensionless stream function, Eq. (7a)
h	film thickness [m]
Nu_x	local Nusselt number, Eq. (13)
Pr	Prandtl number, ν/κ
q	heat flux, $-\rho c_p \kappa\ \partial T/\partial y$ [J s^{-1} m^{-2}]
Re_x	local Reynolds number, Ux/ν
S	unsteadiness parameter, α/b
t	time [s]
T	temperature [K]
u	horizontal velocity component [m s^{-1}]
U	sheet velocity [m s^{-1}]
v	vertical velocity component [m s^{-1}]
x	horizontal coordinate [m]
y	vertical coordinate [m].

Greek symbols

α	constant [s^{-1}]
β	dimensionless film thickness
η	similarity variable, Eq. (7c)
θ	dimensionless temperature, Eq. (7b)
κ	thermal diffusivity [m^2 s^{-1}]
μ	dynamic viscosity [kg m^{-1} s^{-1}]
ν	kinematic viscosity [m^2 s^{-1}]
ρ	density [kg m^{-3}]
τ	shear stress, $\mu\ \partial u/\partial y$ [kg m^{-1} s^{-2}]
ψ	stream function [m^2 s^{-1}].

Subscripts

i	isothermal sheet
o	origin
ref	reference value
s	sheet
x	local value.

sidered by Wang [15] who by means of a similarity transformation reduced the unsteady Navier–Stokes equations to a non-linear ordinary differential equation. The same problem was more recently extended by the present authors [16] to fluids obeying non-Newtonian constitutive equations.

The purpose of the present paper is to explore the heat transfer characteristics of the hydrodynamical problem solved by Wang [15]. It will be demonstrated that exact similarity can be achieved also for the temperature field. Accurate numerical solutions will be provided for two characteristic values of the dimensionless unsteadiness parameter introduced in Ref. [15], covering the range of Prandtl numbers from 0.001 to 1000.

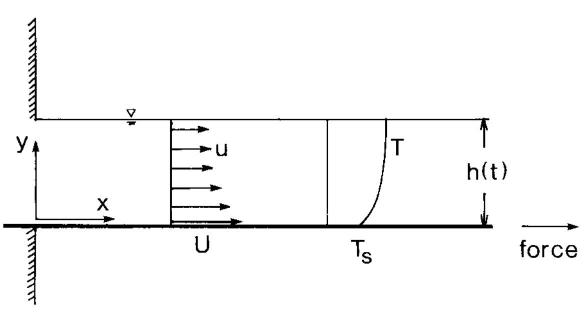

Fig. 1. Schematic representation of a liquid film flow on an elastic sheet.

2. Mathematical formulation

Let us first consider a thin elastic sheet which emerges from a narrow slit at the origin of a Cartesian coordinate system, as shown schematically in Fig. 1. The continuous sheet at $y = 0$ is parallel with the x-axis and moves in its own plane with the velocity.

$$U = bx/(1 - \alpha t) \tag{1}$$

where b and α are both positive constants with dimension time^{-1}. Similarly, the surface temperature T_s of the stretching sheet varies with the distance x from the slit as

$$T_s = T_o - T_{ref}[bx^2/2\nu](1 - \alpha t)^{-3/2}, \tag{2}$$

where T_o denotes the temperature at the slit and $bx^2/\nu(1-\alpha t)$ can be recognized as a local Reynolds number based on the surface velocity U. T_{ref} can be taken either as a constant reference temperature or a constant temperature difference. In the present problem with only the slit temperature T_o being kept constant, T_{ref} could conveniently be set equal to T_o. The special case of an isothermal sheet with $T_s = T_o$, i.e. $T_{ref} = 0$, will be treated separately in Appendix A.

The expression (1) for the sheet velocity $U(x, t)$ reflects that the elastic sheet, which is fixed at the origin, is stretched by applying a force in the positive x-direction. The effective stretching rate $b/(1-\alpha t)$ increases with time since $\alpha > 0$. Similarly, the expression (2) for the temperature $T_s(x, t)$ of the sheet represents a situation in which the sheet temperature decreases from T_o at the slit in proportion to x^2 and such that the amount of temperature reduction along the sheet increases with time. The particular form of the above expressions for $U(x, t)$ and $T_s(x, t)$ has been chosen in order to be able to devise a new similarity transformation which transforms the governing partial differential equations for heat and momentum into a set of ordinary differential equations, thereby facilitating the exploration of the effects of the controlling parameters. It should be noticed, however, that the expressions (1) and (2), on which the following analysis is based, are valid only for time $t < \alpha^{-1}$.

A thin liquid film of uniform thickness $h(t)$ lies on the horizontal sheet (cf Fig. 1). The fluid motion within the film is caused solely by the stretching of the elastic sheet. The velocity field and temperature in the constant-property Newtonian fluid layer are governed by the two-dimensional boundary layer equations for mass, momentum and thermal energy:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{4}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\frac{\partial^2 T}{\partial y^2}, \tag{5}$$

where viscous dissipation of energy has been assumed negligible. The pressure is constant in the surrounding gas phase and the gravity force gives rise to a hydrostatic pressure variation in the liquid film. The associated boundary conditions are

$$u=U;\quad v=0;\quad T=T_s \quad \text{at}\, y=0 \tag{6a}$$

$$\partial u/\partial y=\partial T/\partial y=0 \quad \text{at}\, y=h \tag{6b}$$

$$v=\mathrm{d}h/\mathrm{d}t \quad \text{at}\, y=h \tag{6c}$$

Here, it is implicitly assumed that the mathematical problem is defined only for $x \geq 0$. It is moreover assumed that the surface of the planar liquid film is smooth and free of surface waves. The influence of interfacial shear due to the quiescent atmosphere is negligible and so is the surface tension. The viscous shear stress $\tau=\mu\ \partial u/\partial y$ and the heat flux $q=-\rho c_p\kappa\ \partial T/\partial y$ vanish at the adiabatic free surface, cf Eq. (6b), whereas Eq. (6c) imposes a kinematic constraint on the fluid motion.

Let us now introduce new dimensionless variables f and θ and the similarity variable η:

$$\psi=\{\nu b(1-\alpha t)^{-1}\}^{1/2}xf(\eta) \tag{7a}$$

$$T=T_o-T_{\mathrm{ref}}[bx^2/2\nu](1-\alpha t)^{-3/2}\theta(\eta) \tag{7b}$$

$$\eta=(b/\nu)^{1/2}(1-\alpha t)^{-1/2}y \tag{7c}$$

where $\psi(x, y)$ is the physical stream function which automatically assures mass conservation (3). The velocity components are readily obtained as:

$$u=\partial\psi/\partial y=bx(1-\alpha t)^{-1}f'(\eta) \tag{8a}$$

$$v=-\partial\psi/\partial x=-\{\nu b(1-\alpha t)^{-1}\}^{1/2}f(\eta) \tag{8b}$$

The mathematical problem defined in Eqs. (3)–(6) transforms exactly into a set of ordinary differential equations and their associated boundary conditions:

$$S\left(f'+\frac{\eta}{2}f''\right)+(f')^2-ff''=f''' \tag{9}$$

$$Pr[(S/2)(3\theta+\eta\theta')+2\theta f'-\theta' f]=\theta'' \tag{10}$$

$$f'(0)=1;\quad f(0)=0;\quad \theta(0)=1 \tag{11a}$$

$$f''(\beta)=0;\quad \theta'(\beta)=0 \tag{11b}$$

$$f(\beta)=S\beta/2 \tag{11c}$$

where $S\equiv\alpha/b$ is a dimensionless measure of the unsteadiness and a prime indicates differentiation with respect to η. Moreover, β denotes the value of the similarity variable η at the free surface so that Eq. (7c) gives $\beta=(b/\nu)^{1/2}(1-\alpha t)^{-1/2}h$ for $y=h$. Since β is a yet unknown constant, which should be determined as an integral part of the boundary-value problem, we obtain

$$\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{\alpha\beta}{2}\left(\frac{\nu}{b}\right)^{1/2}(1-\alpha t)^{-1/2} \tag{12}$$

for the rate-of-change of the film thickness. The kinematic constraint (6c) at $y=h(t)$ thus transforms into the free surface condition (11c).

It is noteworthy that the momentum boundary layer problem defined by the ODE (9) subject to the relevant boundary conditions (11) is decoupled from the thermal boundary layer problem, while the temperature field $\theta(\eta)$ is on the other hand coupled to the velocity field. The hydrodynamical problem was solved numerically by Wang [15] for several values of S in the range $0 < S < 2$ and supplemented by asymptotic solutions for thin $(S \approx 2)$ and thick $(S \approx 0)$ films. More

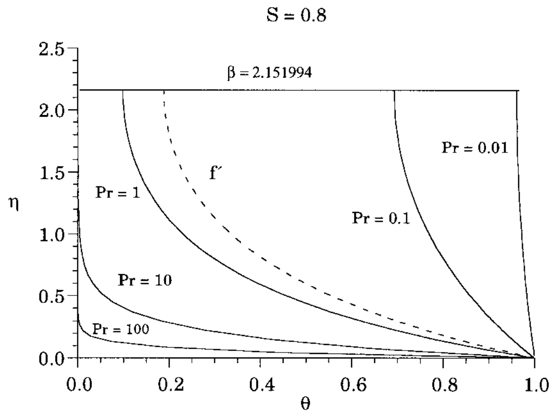

Fig. 2. Dimensionless temperature profiles $\theta(\eta)$ (solid lines) and velocity profile $f'(\eta)$ (broken line) for $S = 0.8$ ($\beta = 2.15199$ and $f''(0) = -1.24581$).

recently, this problem was recovered as a special case of a power-law film flow on an unsteady stretching sheet [16].

3. Numerical results

The non-linear differential equations (9) and (10) subject to the boundary conditions (11) constitute a two-point boundary-value problem, which can be solved by the multiple shooting subroutine MUSN, cf Ascher et al. [17].

Converged numerical results were obtained for two representative values of the unsteadiness parameter S and for Prandtl numbers in the range from 0.001 to 1000. Temperature similarity profiles $\theta(\eta)$ for $S = 0.8$ and $S = 1.2$ are shown in Figs. 2 and 3, respectively. The variation of the free-surface temperature $\theta(\beta)$ with

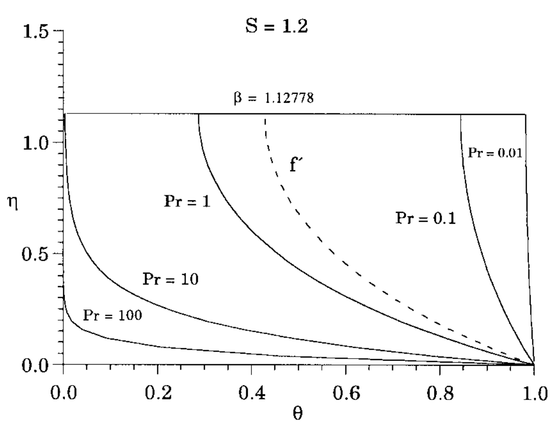

Fig. 3. Dimensionless temperature profiles $\theta(\eta)$ (solid lines) and velocity profiles $f'(\eta)$ (broken line) for $S = 1.2$ ($\beta = 1.12778$ and $f''(0) = -1.27917$).

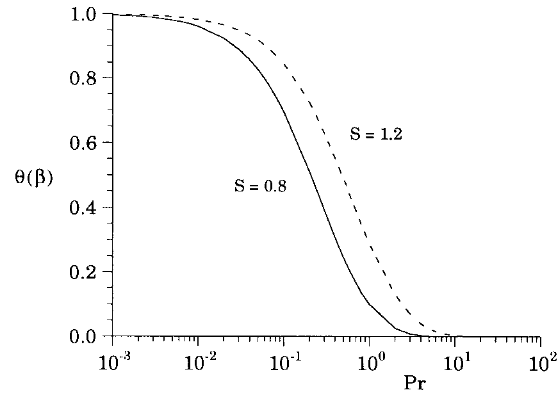

Fig. 4. Dimensionless surface temperature $\theta(\beta)$ vs Prandtl number for $S = 0.8$ (solid line) and $S = 1.2$ (broken line).

Pr is presented in Fig. 4, while the temperature gradient $\theta'(0)$ at the stretching sheet is displayed in Fig. 5. The latter quantity is of particular importance since the heat transfer between the surface and the fluid is conventionally expressed in dimensionless form as a local Nusselt number

$$Nu_x \equiv -\frac{x}{T_{\mathrm{ref}}}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{1}{2}(1 - \alpha t)^{-1/2}\theta'(0)Re_x^{3/2} \qquad (13)$$

where $Re_x = Ux/\nu$ is a local Reynolds number based on the surface velocity U defined in Eq. (1) and T_{ref} denotes the same reference temperature (or temperature difference) as in Eq. (2).

4. Discussions

Let us first recall the major findings of Wang [15] for the hydrodynamic part of the problem. For positive values of the unsteadiness parameter S, Wang observed that solutions exist only for $0 \le S \le 2$. Moreover, when S tended to zero the solution approached the analytical solution due to Crane [5] for an infinitely thick layer of fluid, i.e. $\beta \to \infty$. On the other hand, the limiting solution as $S \to 2.0$ represents a liquid film with infinitesimal thickness, i.e. $\beta \to 0$. Wang observed that the film thickness β decreased monotonically when S was increased from 0 to 2. Since the fluid motion is driven solely by the stretching sheet, the surface gradient $f''(0)$ of the velocity component u parallel to the sheet is negative. It was observed that the magnitude of $f''(0)$ increased with S until it reached a maximum of 1.283 near $S = 1.11$ and thereafter rapidly decreased to zero as S approached 2.0.

The similarity solutions for the dimensionless temperature in Figs. 2 and 3 show that θ decreases monotonically with η, i.e. with the distance from the elastic

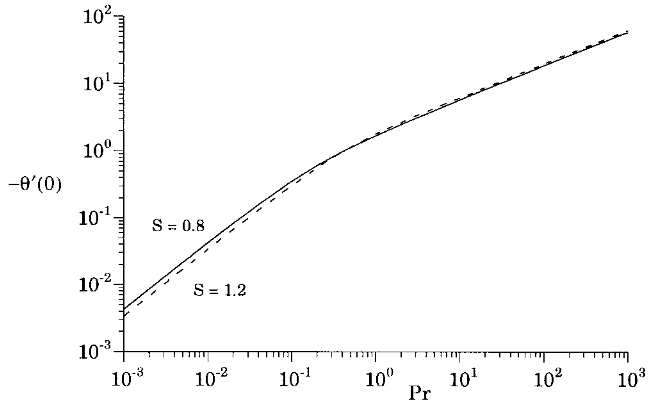

Fig. 5. Dimensionless heat flux $-\theta'(0)$ at the sheet vs Prandtl number for $S = 0.8$ (solid line) and $S = 1.2$ (broken line).

sheet, for all Prandtl numbers. This implies that the temperature T gradually increases with η from $T_s(\leq T_o)$ at $\eta = 0$. It is noteworthy that θ vanishes at the free surface $\eta = \beta$ for sufficiently high Prandtl numbers, i.e. $Pr \gtrsim 10$, which means that the surface temperature T equals the temperature T_o which prevails at the origin $x = y = 0$. This in turn implies that the surface temperature becomes independent both of position x and time t. The variation of $\theta(\beta)$ with Pr displayed in Fig. 4 reveals that uniformity of the surface temperature can be achieved at somewhat lower Prandtl numbers for $S = 0.8$ than for $S = 1.2$. At high Prandtl numbers, the thermal boundary layer is contained within the lower part of the liquid film and the temperature gradients vanish adjacent to the free surface. This is fully analogous to the situation in a steady-state aerodynamic boundary layer in an infinite fluid medium at high Pr. The reason why the adjustment of the temperature from T_s to T_o takes place over a thin thermal boundary layer is the gradually increasing importance of the terms on the left hand side of Eq. (10) with increasing Pr. As the thickness of the thermal boundary layer reduces at higher Pr, the isothermal layer with uniform temperature T_o, but variable velocity, on top of the thermal boundary layer gradually thickens. In this isothermal liquid layer, a delicate balance between thermal convection and local variations prevails, while thermal conduction is negligible. In the limit $Pr \to \infty$, the local heat transfer rate at the stretching sheet is therefore controlled by the velocity in the immediate vicinity of the sheet which can be expanded as $f'(\eta) \simeq 1 + \eta f''(0)$ for $\eta \ll 1$. The magnitude of the velocity gradient at the sheet $f''(0)$ exhibited only a modest variation over the parameter range from $S = 0.8$ to $S = 1.2$ with a local maximum at $S = 1.11$ [15]. The observation that $-f''(0)$ for $S = 1.2$ is less than 3% higher than for $S = 0.8$ explains why the dimensionless heat flux $-\theta'(0)$ for $S = 1.2$ only marginally exceeds that for $S = 0.8$ for $Pr > 1$ in Fig. 5.

For Prandtl numbers of order unity and below the surface temperature $\theta(\beta)$ attains a finite value below 1 (see Fig. 4) and the temperature gradients extend all the way to the free surface. In the limiting case $Pr \to 0$, however, the dimensionless surface temperature tends to unity, i.e. the temperature T becomes uniform in the vertical direction and equals T_s. This is consistent with the trivial solution $\theta(\eta) = 1$ obtained from the thermal energy equation (10) when $Pr = 0$. For small but not zero values of Pr, the temperature gradient θ' can be neglected in Eq. (10). If the streamwise velocity profile for simplicity is approximated as $f'(\eta) \approx 1$, the asymptotic solution

$$\theta(\eta) \cong [e^{(2\beta-\eta)c} + e^{c\eta}]/[1 + e^{2\beta c}] \tag{14a}$$

where

$$c = [Pr(2 + 3S/2)]^{1/2} \ll 1 \tag{14b}$$

can be derived analytically. The gradient of the temperature profile at the stretching surface $y = 0$ thus becomes

$$\theta'(0) \approx -c^2\beta \tag{15}$$

in the high diffusivity (low Pr) limit. The combined parameter c^2 increases with S, but this effect on the temperature gradient (15) is more than outweighed by the observation by Wang [15] that β is a rapidly decaying function of S. The most striking observation from Fig. 5 is, however, the linear variation of $\theta'(0)$ with Pr for $Pr \ll 1$, i.e. fully in accordance with the asymptotic solution (15).

5. Concluding remarks

The purpose of this paper was to present an exact similarity solution for momentum and heat transfer in an unsteady liquid film whose motion is caused solely by the linear stretching of a horizontal elastic sheet. The main findings can be summarized as follows:

1. A new similarity solution for the temperature field has been devised, which transforms the time-dependent thermal energy equation to an ordinary differential equation.
2. The Prandtl number Pr and the dimensionless parameter $S=\alpha/b$, the latter which reflects the relative importance of unsteadiness α to the stretching rate b, were identified as the only controlling parameters.
3. The temperature was observed to increase monotonically from the elastic sheet towards the free surface, except in the high diffusivity limit $Pr \rightarrow 0$ in which the surface temperature approached the variable sheet temperature. At sufficiently high Pr, on the other hand, the surface temperature became equal to T_{o} and thus independent both of position and time.
4. The influence of the unsteadiness parameter S on the heat flux from the liquid film to the stretching sheet was more pronounced at low Prandtl numbers than for $Pr \gtrsim 1$, whereas the surface temperature was most affected by S at intermediate Prandtl numbers.

Appendix A

The special case of an isothermal sheet $T_{\mathrm{s}}=T_{\mathrm{o}}$, i.e. $T_{\mathrm{ref}}=0$, must be treated separately. The dimensionless temperature $\theta(\eta)$ defined in Eq. (7b) does no longer apply and should be replaced with a new dependent variable θ_i defined as

$$T = T_{\mathrm{o}}\theta_i(\eta) \tag{A1}$$

which should attain the same values as θ at the boundaries $\eta=0$ and $\eta=\beta$, cf Eq. (11). The thermal energy Eq. (5) now transforms into the following ODE

$$Pr\left[\frac{S}{2}\eta - f\right]\theta_i' = \theta_i'' \tag{A2}$$

for which the trivial solution $\theta_i(\eta)=1$ applies for all Prandtl numbers. Physically this solution represents an insulating liquid film with uniform temperature $T=T_{\mathrm{o}}$.

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